**CMSI 282 Problem Set #7**
**Due April 28[1], 2009**

1) Consider this procedure for dividing $X$ among $A$, $B$, and $C$:

   Step 1: $A$ is instructed to cut $X$ into two pieces, $X_1$ and $X_2$, such that $\mu_A(X_1) = 1/3$ and $\mu_A(X_2) = 2/3$.

   Step 2: $B$ is then instructed to cut $X_2$ into pieces $X_{21}$ and $X_{22}$, so that $\mu_B(X_{21}) = \mu_B(X_{22})$.

   Step 3: The three players then select their pieces from $\{X_1, X_{21}, X_{22}\}$ in the order $C$, then $A$, and, finally, $B$.

   Does this procedure accomplish simple fair division for the three players? Prove it.

2) Jessica's Algorithm: $A$, $B$, and $C$ wish to divide a cake so that $A$ gets $1/2$, $B$ gets $1/3$, and $C$ gets $1/6$. Here is a possible procedure for accomplishing this division:

   Step 1: Instruct $A$ and $B$ to play cut-and-choose, so that each receives half of the cake according to his/her own evaluation.

   Step 2: Instruct $A$ and $B$ to divide their shares into equivalent thirds.

   Step 3:Give $C$ her choice of these six pieces; she exits the game.

   Step 4: If $C$ chose one of $B$'s pieces, there is nothing more to be done; otherwise ($C$ must have chosen one of $A$'s pieces, so) $A$ gets to choose one of $B$'s pieces.

   Does this procedure accomplish simple fair division for the three players? Prove it.

3) Alice and Bob wish to divide a cake in the ratio $7:3$, i.e., Alice should receive at least $70\%$ of the cake, according to her valuation, and Bob should receive at least $30\%$ of the cake, according to his valuation. Accordingly, they decide to use the following procedure:

   Step 1: Bob is instructed to cut the cake into six pieces in the ratio $3:2:2:1:1:1$.

   Step 2: Alice is instructed to mark those pieces that, in her opinion, have been "appropriately cut" by Bob.

   Step 3: If Alice has identified pieces from which some subset adds up to $7/10$ (according to her valuation), then she takes that subset and gives Bob the rest;

[1] N.B., Revised due date.

otherwise, from the unmarked pieces, Bob chooses a subset that adds up to *3/10* (according to his valuation when he cut the cake).

a) Does this procedure work? If so, prove it; if not, why not?

b) Will this procedure work if Bob is instructed to cut six pieces in the ratio *3:3:1:1:1:1*? Why, or why not?

4) Alice, Bob, and Charles (also known as *A*, *B*, and *C*) wish to divide a cake in the ratio *A:B:C = 7:3:2,* i.e., Alice should receive at least *7/12* of the cake (according to her valuation), etc. How can they accomplish this? Give an algorithm.

5) *A*, *B*, and *C* wish to divide a cake using simple fair division. *A* was instructed to cut the cake into three equivalent pieces ($X_1$, $X_2$, and $X_3$), then *B* and *C* were told to identify pieces that they felt were at least one-third of the cake. They (B and C) identified <u>only</u> piece $X_1$ . What should they do?

6) Kuhn's Lone Divider: *A*, *B*, and *C* wish to divide *X* (using simple fair division). Accordingly, one of the players is instructed to cut *X* into equivalent pieces, $X_1$, $X_2$, and $X_3$. Each player then indicates which piece(s) are acceptable to her. The information might be summarized in a table like this (with asterisks and dashes indicating, respectively, acceptable versus unacceptable pieces):

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| *A* | * | * | * |
| *B* | * | * | - |
| *C* | * | * | - |

a) In general, will every row have at least one asterisk?
b) In general, will there always be a row with three asterisks?
c) Show, for the table above, that fair division is possible.
d) Show, for the table below, that fair division is possible.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| *A* | * | * | * |
| *B* | * | - | - |
| *C* | * | - | - |

e) Prove that, for any (valid, 3-by-3) matrix, a fair division is possible.